Network Algorithms

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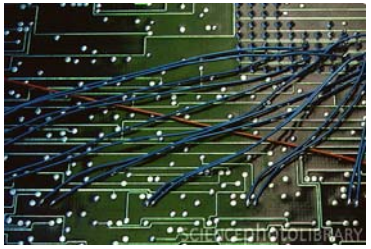
The plan

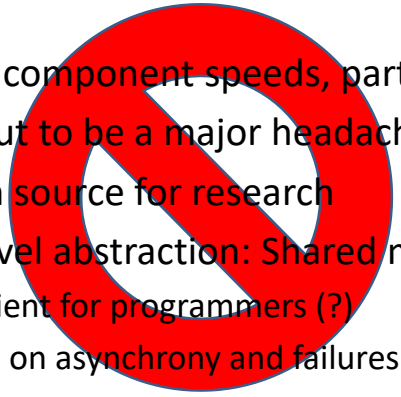
- Intro: LOCAL model, synchronicity, Bellman-Ford
- Subdiameter algorithms: independent sets and matchings
- CONGEST model: pipelining, more matching, lower bounds

Distributed Algorithms

- Turing's vision: multiple heads, multiple tapes, but central control
- Today's technology: hook up components by communication lines
- Abstraction: network of processors exchanging messages



Some Issues



- Different component speeds, partial failures
- Turned out to be a major headache…
- … = a rich source for research
- Higher level abstraction: Shared memory
  - Convenient for programmers (?)
  - Focuses on asynchrony and failures

Not our topic!

Our Focus: Communication

The **LOCAL** model

- Connectivity ≡ a graph $G = (V, E)$
- $|V| = n, |E| = m$, diameter = D
- Nodes compute, send & receive msgs
- No failures (nodes or links)
- Running time:
  - DEFINE: longest message delay ≡ one time unit
  - Neglect local processing time

The **LOCAL** model: Typical Tasks

- Compute functions of the topology
  - Spanning Trees: Breadth-First (shortest paths), Minimum weight (MST)
  - Maximal Independent Set, Maximal Matching
- Communication tasks
  - Broadcast, gossip, end-to-end
- In general: **input/output relation**
  - Dynamic version: **reactive tasks**

The locality of distributed problems

A problem can be solved in T time in **LOCAL** iff the following algorithm works:

- Each node collects full information from its T-neighborhood, and computes output

Time is the "radius" of input influence.
LOCAL time = problem locality!

- Example: $\Omega(\sqrt{\log n / \log\log n})$ lower bound on time for approx. MIS, Matching [KMW'06]

Example: Broadcast

- A source node has a message all nodes need to learn
  - Input: environment to source at time 0
- Protocol: When a new message received, send to all neighbors (**flooding**)
- Time: $O(D)$
- Can be used to build a spanning tree:
  - Mark edge delivering first message as parent

Example: Find maximum

- Each node has an input number from $[1, M]$
- Want to find maximum
- Protocol: Whenever a new maximum discovered, announce to all neighbors
- Time to convergence: $O(D)$
- #messages: $O(nm)$
- Message size: $O(\log M)$ bits

Example: What's n?

- Goal: find the number of nodes in the system
- Must assume some symmetry breaking
  - exercise
- Standard assumption: Unique IDs from $[1, N]$
  - Usually assume $N \in n^{O(1)}$
- Solution:
  - Use a broadcast from each node
- Converge in $O(D)$ time
- #messages $O(mn)$

Example: What's n?

Another symmetry breaking tool: randomization

Algorithm:

1. Each node v chooses a value $x_v \sim G\left(\frac{1}{2}\right)$.
2. Find $\max\{x_v \mid v \in V\}$
3. Output 2^{max}.

Message length: $O(\log \log n)$.

Can be off by a $\log n$ factor. Repeat and report average to decrease variance.

Generic Algorithm

1. Each node broadcasts its input
2. Each node computes locally its output

- Time: $O(D)$, #messages $O(mn)$

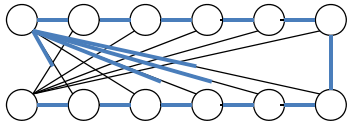
Can do better!

1. Build a single tree (how?)
2. Apply algorithm in tree

- #messages $O(n^2)$
- Time?

Asynchrony gives trouble

- A tree may grow quickly in a skewed way…



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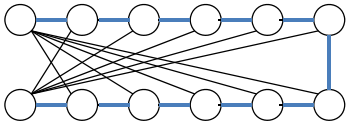

…But when used for the second time, we may pay for the skew!

The Bellman-Ford Algorithm

- Goal: Given a root, construct a shortest-paths tree
- Protocol:
  - every node v maintains a variable d_v
  - Root r has $d_r = 0$ always
  - Non-root v sets $d_v = \min\{d_u + 1 : (u,v) \in E\}$
- Can show: stabilizes in time $O(D)$

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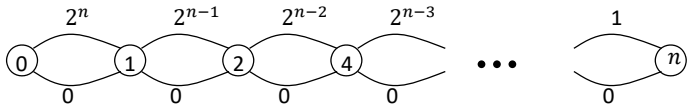
B-F: Trouble With Asynchrony

Convergence in $O(D)$ time, but strange things can happen…



- Schedule S_0: empty schedule
- Schedule S_{i+1}:
  1. Allow one message: from $n-i-1$ to $n-i$ on edge of weight 2^i (incoming value: $d_{n-i-1}+2^i$)
  2. Apply S_i nodes to nodes $n-i, \dots n$
  3. Allow another message: from $n-i-1$ to $n-i$ on edge of weight 0 (incoming value: d_{n-i-1})
  4. Apply S_i nodes to nodes $n-i, \dots n$

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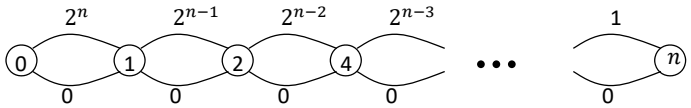


- Under schedule S_n: node i receives 2^i distinct messages
- Node n receives 2^n messages in n time units?!

Synchronous model

If all processors and links run in the same speed:

Execution consists of **rounds**. In each round:

1. Receive messages from previous round
   - Round 1: receive inputs
2. Do local computation
3. Send messages

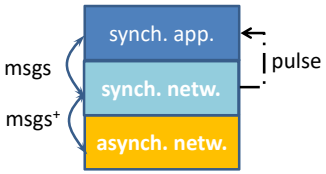
Avoid skewed evolution!

Synchronous BFS tree construction

- Protocol: When first message received, mark origin as parent and send to all neighbors
  - Input: environment to source at time 0
  - Break ties arbitrarily
- Natural uniform “ball growing” around origin
- Time: $O(D)$

Synchronizer

- Can emulate synchronous environment on top of asynchronous one
- Abstraction: consistent **pulse** at all nodes



How:

- send a message to each neighbor in each round (send null message if nothing to send)
- Emit pulse when received messages from all neighbors

Therefore:

- Asynchronous networks are interesting only if there may be faults
  - Or when we care about #messages
- We henceforth assume synchronous networks…
  - But we need to account for messages!

Generic Synchronous Algorithm

- **Any** i/o-relation solvable in diameter time:
  1. Construct a BFS tree
     (need IDs/randomization to choose root: Leader Election!)
  2. Send all inputs to root (“convergecast”)
  3. Root computes all outputs, sends back (“broadcast”)
- Ridiculous? That’s the client-server model!
  - Bread-and-butter distributed computing in the 70’s-90’s, and beyond…
- Interesting? Theoretically : sub-diameter upper and lower bounds

Subdiameter algorithms

Independent Sets

- Independent set (IS): a set of nodes, no neighbors
- Maximum: **terribly** hard
- Maximal: cannot be extended
  - Can be MUCH smaller than maximum IS
- Trivial sequentially!
  - Linear time
- Can this parallelized?

Independent Sets

Turan's theorem: There always exists an IS of size $n/(\bar{d}+1)$. [$\bar{d} = 2m/n$ is the average degree]

Proof: By the **probabilistic method**.
Assign labels to nodes by a random permutation.
Let IS be the nodes whose label is a *local minimum*.

Lemma: $E[|\mathrm{IS}|] \geq \frac{n}{\bar{d}+1}$. ●

Lemma & Proof

Lemma: $E[|\mathrm{IS}|] \geq \frac{n}{\bar{d}+1}$.

Proof: By the Means Inequality.

- $E[|\mathrm{IS}|] = \sum_v \frac{1}{d(v)+1} = n \cdot \mathrm{AM}\left\{\frac{1}{d(v)+1}\right\}$
  $\geq n \cdot \mathrm{HM}\left\{\frac{1}{d(v)+1}\right\}$
  $= n \cdot \frac{n}{\sum_v (d(v)+1)}$
  $= \frac{n}{\bar{d}+1}$. ●

$$\mathrm{AM}(S) = \sum_{x \in S} x/|S|$$

$$\mathrm{HM}(S) = \frac{|S|}{\sum_{x \in S} 1/x}$$

$$\mathrm{AM}(S) \geq \mathrm{HM}(S)$$

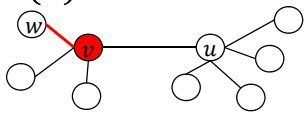
Distributed MIS algorithm

- Each node v chooses a random label $\ell(v)$
  - Say, in $[1, N^{2+c}]$ to avoid collisions w.h.p.
- Local minima enter MIS, neighbors eliminated
- Repeat.

Claim: In expectation, at least half of the **edges** are eliminated in a round.

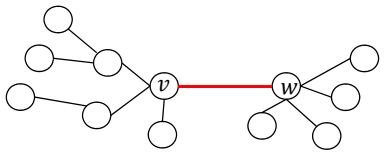
Proof of Claim

Say node v is **killed** by neighbor u if $\ell(u)$ is smallest in $N(v) \cup N(u)$.



An edge (v, w) is killed by node u if v is killed by u.

Observation: An edge (v, w) can be killed by at most two nodes: the node with minimal label in $N(v)$ and the node with minimal label in $N(w)$



Proof of Claim (cont.)

$$\Pr[v \text{ killed by } u \mid (u,v) \in E] \geq \frac{1}{d(v)+d(u)}.$$

Hence: $E[\#\text{edges killed}] \geq$

$$\frac{1}{2}\sum_u E[\#\text{edges killed by } u]$$

$$\geq \frac{1}{2}\sum_{v \in V} d(v) \sum_{u \in N(v)} \frac{1}{d(u)+d(v)}$$

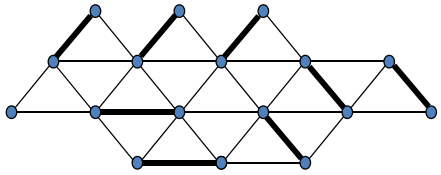
$$= \frac{1}{2}\sum_{(u,v)\in E} \left(\frac{d(v)}{d(u)+d(v)} + \frac{d(u)}{d(u)+d(v)} \right)$$

$$= \frac{1}{2}|E|.$$

Distributed Matching

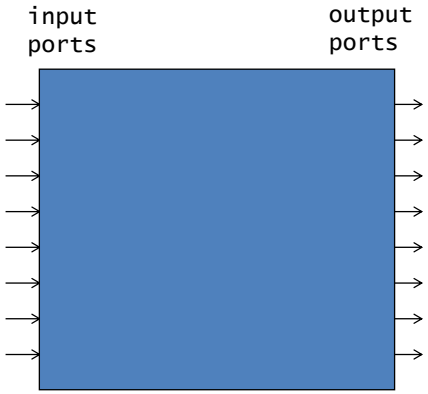
Definitions

- Input: Graph G=(V,E), with weights w : E⟶R⁺
- A matching: a set of disjoint edges
- Maximum cardinality matching (MCM)
- Maximum weighted matching (MWM)



Application: Switch Scheduling

Goal: move packets from inputs to their outputs



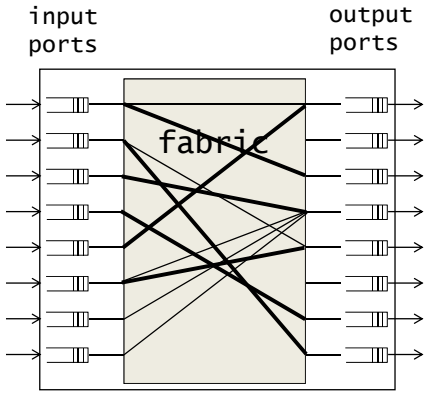
Application: Switch Scheduling

Goal: move packets from inputs to their outputs

At each time step, fabric can forward

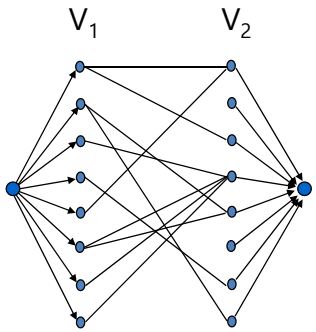
- one packet from each input
- one packet to each output

- To maximize throughput, find MCM!



Note: Bipartite Graphs

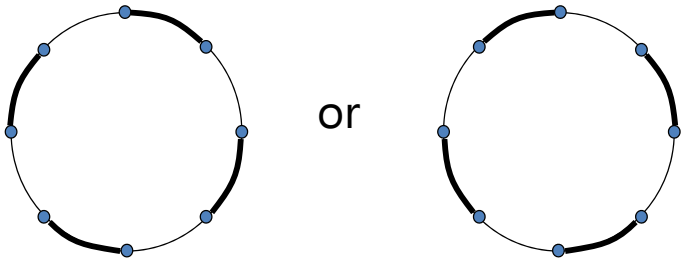
- In many applications, nodes are partitioned into two subsets (input/output, boys/girls)
- Bipartite graphs: $G = (V_1 \cup V_2, E)$ where $V_1 \cap V_2 = \emptyset$ and $E \subseteq V_1 \times V_2$
- Matching is simpler in this case
  - Bipartite MCM: max flow
  - bipartite MWM: min cost flow



Distributed Matching

Clearly, MCM must take diameter time!

- Information traverses the system: must decide between the following alternatives



MCM: Reduction to MIS

- **Edge graph:** If $G = (V, E)$ then in $EG(G)$:
  - The node set is E
  - (e, e') are connected in $EG(G)$ if they share a node in G
- Observation: M is a matching in $G \Leftrightarrow$ M is independent in $EG(G)$

Approximate Distributed Matching

Theorem: Maximal matching is ½-approximate MCM.

Proof: Let M be a maximal matching, and M^* an MCM.

Observe that

1. Any edge in M touches ≤ 2 edges in M^*.
2. Any edge in M^* touches ≥ 1 edge in M.

Map each edge in M to the M^* edges it touches.

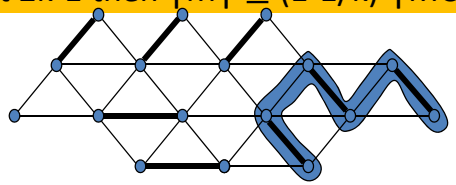
By (1): #edges mapped to is at most $2|M|$.

By (2), all MCM edges are mapped to, i.e., $|M^*| \leq 2|M|$. ●

Augmenting Paths

- Basic concept: augmenting path for a matching M
  - alternating M-edges and non-M edges
  - starts and ends with unmatched nodes
- Flipping membership in M increases the size of matching

Theorem: if all augmenting paths w.r.t. M have length at least 2k-1 then $|M| \geq (1-1/k) \cdot |MCM|$



MCM: Generic Approximation

Generic algorithm: input is G, ε

M = ∅

For k=1 to 1/ε do

  Create "**conflict graph**" CG(k):

    nodes = augmenting paths of length 2k-1

    edges = pairs of intersecting augmenting paths

  Find MIS in CG(k)

  Augment M by flipping edges of paths in MIS //well defined!

MCM: Generic Approximation

Why works? Hopcroft-Karp.

Distributed implementation:

1. Nodes collect map of neighborhood to distance 2k+1
2. Appoint **leader** for each AP (say, endpoint with smaller ID)
3. Leaders simulate MIS algorithm of conflict graph

Complexity:

- Time: $O(k)$ rounds for $(1-1/k)$-approximation
- Messages are large... (neighborhood to distance 2k+1)
- #paths is $n^{O(k)} \Rightarrow$ much computation, even larger messages

Ergo: The **CONGEST** model

- Same as **LOCAL**, but ***messages may contain up to B bits***
- Usually $B = O(\log n)$
  - Allows messages to carry $O(1)$ IDs and variables of magnitude $O(n^k)$
  - Similar to the "word model" in RAM
  - Exact value of B usually doesn't matter
- Captures network algorithms more faithfully

Canonical Algorithm for **CONGEST**

Same algorithm:

1. Construct a BFS tree
2. Send all input to root
3. Root computes all outputs, sends back

Running time: $O(D + \#\text{inputs} + \#\text{outputs})$.

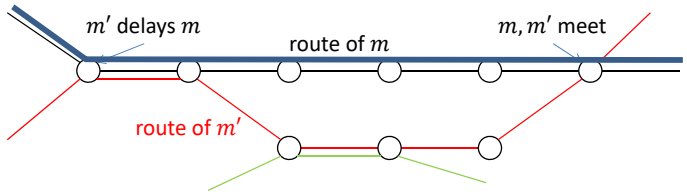
Why? In CONGEST, messages may be delayed!

- **Pipelining**…

Basic Pipelining

Theorem: Suppose k messages travel on **shortest paths**. Then no message is delayed more than $k-1$ steps. (Any starting times!)

- Proof: Place a token on a message m. If it is **delayed** by m' **and then meets** m' again, let token take detour with m'.



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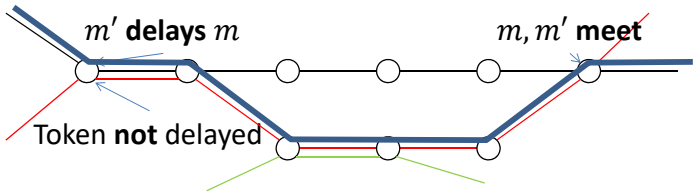


Proof (cont.)

- Before and after token's detour:
  - Same endpoints, same start time, same finish time
  - Same length (shortest paths!)
  - Hence same number of delays
- But the delay at switching point eliminated
- Consider a sequence of detours: the vector of times-of-delay decreases **lexicographically**
- Hence if we repeatedly switch, process ends

Proof (end)

- Applying detour maintains #delays
- Eventually cannot apply detour:
  - Recall: Detour applicable if token is **delayed** by m' **and then meets** m' again.
- If detour not applicable, no message delays token twice
- Final message does not delay token at all.
- Hence #delays $\leq$ # other messages. ●

Canonical Algorithm for **CONGEST**

Same algorithm:

1. Construct a BFS tree
2. Send all input to root
3. Root computes all outputs, sends back

Running time: $O(D + \#\text{inputs} + \#\text{outputs})$.
Why?

- In a tree, all paths are shortest, length $O(D)$.
- Each piece of input is a message

Back to Matching

Generic algorithm: input is G, $\epsilon > 0$

M = ∅

For k:=1 to $\lceil \frac{1}{\epsilon} \rceil$ do

Create "**conflict graph**" CG(k):

nodes := augmenting paths of length 2k-1

edges := pairs of intersecting augmenting paths

Find MIS in CG(k)

Augment M by flipping edges of paths in MIS //well defined!

Matching in CONGEST?

Recall MIS: random label per AP, local minima win

Observations:

- Leader can select its own local winner
- Winner can be constructed rather than discovered
- All that's needed is that each node knows how many APs it belongs to

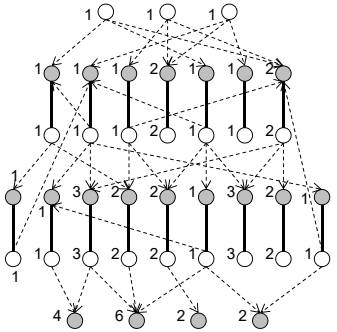
And that's easy in bipartite graphs!

Counting APs in Bipartite Graphs

Goal: count how many shortest APs of prescribed length end at gray nodes.

Idea: BFS.

- start with unmatched white nodes (1)
- each node sums all first incoming numbers
  - later messages ignored
- white nodes send sum to all neighbors
- gray nodes send sum to their mate
- last nodes know exactly how many APs end with them



Counting APs in Bipartite Graphs

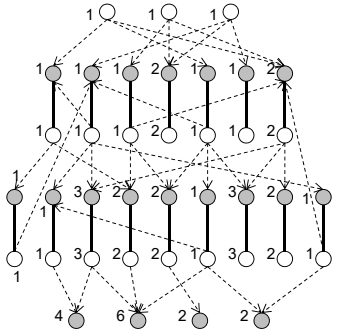
Goal: pick a uniformly random AP among all ending at a specific node.

Idea: inductive construction.

- start at leader (bottom grey)
- at grey nodes, pick edge with probability proportional to number on its far end
- at white nodes, follow matching edge

This defines a uniformly chosen random winner path for each leader.

Remains: resolve conflicts between leaders.



Algorithm for Bipartite Graphs

- Count number of augmenting paths for each leader
- A leader v of m paths picks a number w_v distributed like the minimum of m uniform variables (easy).
- Token selects next edge with probability proportional to #paths that lead to that edge.
- Each node records the smallest w_v it has seen
- After creating path, token backtracks unless killed
- best path joins MIS, etc.

Algorithm for Bipartite Graphs

- #nodes in conflict graph = #APs of length k
  - at most $n\Delta^{k/2} \leq n^{O(k)}$.
- Hence
  - Random labels have $O(k \log n)$ bits (k messages)
  - #iterations is $O(k \log n)$
- Iteration k is emulated in $O(k)$ steps
- Time complexity: $O(k^2 \log n)$ for a given k, $O(\varepsilon^{-3} \log n)$ over all since $k = 1,2, \dots, 2/\varepsilon$.

General Graphs

- Idea: reduction to bipartite
- Means:
  - Color nodes black or white randomly
  - Ignore monochromatic edges
  - Apply bipartite algorithm
- Can prove: Repeating $2^{O(1/\varepsilon)}$ times suffice (w.h.p.)

Theorem: For any **constant** $0 < \varepsilon < 1$, in any graph, $(1 - \varepsilon)$-MCM can be computed distributively in time $O(\log n)$ using messages of size $O(\log n)$.

Lower Bounds

Bad Graphs for **CONGEST**

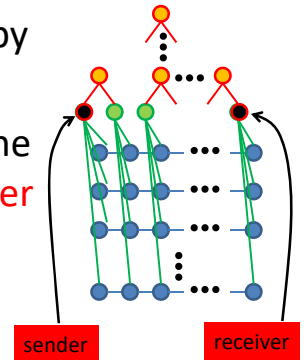
- Low diameter: there's always a shortcut
  - Good enough for **LOCAL**
- In **CONGEST**: when shortcuts are narrow, low diameter not enough to transmit massive data
- State of the art: graphs of diameter $\log n$ for problems that need to transport $\sqrt{n}$ bits
  - Extends to diameter 3 with weaker lower bounds

Bad Graphs for **CONGEST**: Basic Construction

- Bulk: $\sqrt{n}$ paths of length $\sqrt{n}$ each
- Connect corresponding nodes by a star
- Build a tree whose leaves are the star centers => $O(\log n)$diameter

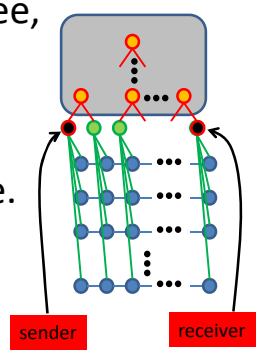
File Transfer Problem: transmit $\sqrt{n}$ bits from sender to receiver

- PR'99: Must take $\Omega(\sqrt{n})$ time!



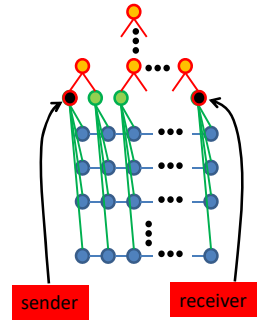
Bad Graphs for **CONGEST**: Very small diameter [LPP'01,E'04]

- For $3 < D \leq \log n + 2$: replace binary tree with a *d*-ary tree, where $d = n^{1/2(D-2)}$.
  Lower bound: $\Omega\left(n^{\left(\frac{1}{2}-\frac{1}{2D-2}\right)}\right)$
- For $D = 3$: replace tree by a clique.
  Lower bound: $\Omega\left(n^{1/4}\right)$
- How about $D = 2?$ $D = 1?!$



Bad Graphs for **CONGEST**: Applications

- To prove a time lower-bound on Π, reduce "file transfer" to Π:
  - Π = MST (edge weights) [PR'99]
  - Stable Marriage (rankings) [KP'09]
- Strengthening: Can't even **approximate** MST



Conclusion

- Simple abstractions to model networks
- Some nice algorithmic techniques
- Many open problems
  - Heterogeneous link bandwidth
  - Complexity of applications in low-diameter networks
  - Incorporating faults into model

Thanks!